Exam for Pd.D. School in Bertinoro 2016 : Distributed Control and its applications

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The final examination consists in providing the solution of one of the following exercises.

EXERCISE 1 : Ratio Consensus

Let us consider the (undirected) communication graph in the figure.

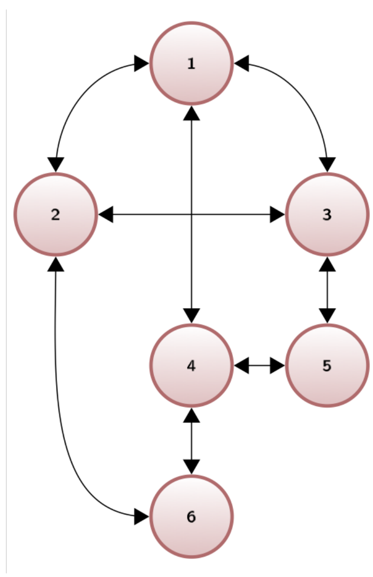

Figure 1:

Design a consensus matrix P which is only *column stochastic* and compatible with the graph, i.e. $P \geq 0, \mathbf{1}^T P = \mathbf{1}^T$ but $P\mathbf{1} \neq \mathbf{1}$, where $\mathbf{1} = [1\,1 \cdots 1]^T$, and $[P]_{ij} = 0$ if $(j,i) \notin \mathcal{G}$. Let us consider the problem of computing the average of some local values $\theta_i, i = 1, \ldots, 6$, i.e. we want to compute in a distributed fashion $\frac{1}{N}\sum_i \theta_i$.

Consider the following distributed algorithm:

$$x(t+1) = Px(t), \quad x_i(0) = \theta_i \tag{1}$$

$$y(t+1) = Py(t), \quad y_i(0) = 1 \tag{2}$$

where y_i are additional local variables.

1. Prove that
$$\lim_{t\to\infty} x_i(t) = \alpha_i$$
$$\lim_{t\to\infty} y_i(t) = \beta_i$$
and find an expression for α_i and β_i

2. Prove that the local quantities
$$z_i(t) := \frac{x_i(t)}{y_i(t)}$$
have the following property:
$$\lim_{t\to\infty} z_i(t) = \frac{1}{N}\sum_i \theta_i, \forall i$$

3. Implement the previous algorithm in Matlab and plot the evolution of all the variables $x_i(t), y_i(t), z_i(t)$.

EXERCISE 2 : Broadcast-based ratio consensus

Assume that at each time step only one node i sends its local variables $x_i(t), y_i(t)$ to its neighbours and that the sending node i and the receiving nodes $j \in \mathcal{N}_i$ updates their local variables as follows:

$$x_i(t+1) = (1 - \delta|\mathcal{N}_i|)x_i(t)$$
$$x_j(t+1) = x_j(t) + \delta x_i(t), \quad \forall j \in \mathcal{N}_i$$

where $|\mathcal{N}_i|$ correspond to the number of neighbours of i, while all other nodes do nothing, i.e.

$$x_\ell(t+1) = x_\ell(t), \quad \forall \ell \notin \{\mathcal{N}_i, i\}$$

Assume that this algorithm is applied using a round-robin protocol, i.e. at time $t = 1$ node 1 sends its local data and the updates defined above are performed, at time $t = 2$ node 2 sends its local data till at at time $t = 6$ node 6 sends its local data, and then the nodes restart, i.e. at time $t = 7$ node 1 sends its local data and so on.

1. Show that at each time step the algorithm can be written as
$$\begin{aligned} x(t+1) &= P(t)x(t), \quad x_i(1) = \theta_i \qquad (3) \\ y(t+1) &= P(t)y(t), \quad y_i(1) = 1 \qquad (4) \end{aligned}$$
where
$$P(t) \in \{P_1, \ldots, P_6\}$$
where $\mathbf{1}P_i = \mathbf{1}, \forall i$. Under the round-robin protocol $P(t) = P_{\text{mod}(t-1,6)+1}$, where $\text{mod}(m, n)$ is the remainder after division of m by n.

2. Find the condition on δ such that the matrices P_i are all column-stochastic.

3. Implement the previous algorithm in Matlab and plot the evolution of all the variables $x_i(t), y_i(t), z_i(t)$ and observe that
$$\lim_{t\to\infty} z_i(t) = \frac{1}{N}\sum_i \theta_i$$
but that $x_i(t)$ and $y_i(t)$ do not converge to a constant value.

4. **(optional)** Let us define

$$Q(t) = P(t-1) \cdot P(1)$$

so that

$$x(t) = Q(t)x(1), \quad y(t) = Q(t)y(1)$$

Show that there exist a time-varying vector $w(t) \in \mathbb{R}^6$ and a positive scalar μ such that $w(t) \geq \mu > 0$, $\mathbf{1}^T w(t) = 1$ and

$$\lim_{t\to\infty} Q(t) - w(t)\mathbf{1}^T = 0$$

In order to prove this result prove the following propositions

(a) Let $S(t) := P^T(t)P^T(t+1)\cdots P^T(t+5)$. Show that $S(t)$ is row-stochastic and that the associated graph $\mathcal{G}_{S(t)}$ is strongly connected for any t.

(b) Prove that there exists T such that $V(t) := P^T(t)P^T(t+1)\cdots P^T(t+T-1)$ has the following property

$$[V(t)]_{ij} \geq \eta > 0, \forall t, \forall i, j$$

where η is a positive constant. This implies that

$$\epsilon_{V(t)} := \max_j \min_i [V(t)]_{ij} \geq \eta > 0, \forall t$$

(c) Let us define the following Lyapunov-like equation:

$$h(t) = \max_i(\max_j [Q(t)]_{ij} - \min_j [Q(t)]_{ij})$$

Show that

$$\lim_{t\to\infty} h(t) = 0$$

is equivalent to

$$\lim_{t\to\infty} Q(t) - w(t)\mathbf{1}^T = 0$$

for some $w(t)$.

(d) Show that

$$h(kT) \leq (1-\eta)^k h(1)$$

(e) Show that there exists an integer $\overline{T}$ and a positive constant μ such that $w(t)$ must satisfy $w(t) \geq \mu > 0$ and $\mathbf{1}^T w(t) = 1$ for all $t \geq \overline{T}$.

(f) Use the previous results to argue that

$$\lim_{t\to\infty} z_i(t) = \frac{1}{N}\sum_i \theta_i, \quad \forall i$$

exponentially fast

EXERCISE 3 : Randomized symmetric gossip consensus

Consider the same communication graph given in the previous figure and number the 8 different communication edges from 1 to 8. We would like to implement a randomized symmetric gossip algorithm, where at each time instant one edge (i, j) is chosen and the two corresponding nodes perform the following updates:

$$x_i(t+1) = \frac{1}{2}x_i(t) + \frac{1}{2}x_j(t)$$

$$x_j(t+1) = \frac{1}{2}x_i(t) + \frac{1}{2}x_j(t)$$

while all other nodes do nothing, i.e.

$$x_\ell(t+1) = x_\ell(t), \forall \ell \notin \{i,j\}$$

Show that the algorithm can be written as:

$$x(t+1) = P(t)x(t), x(0) = \theta$$

where

$$P(t) \in \{P_1, \dots, P_8\}$$

and P_i are symmetric stochastic matrices each associated to a specific communication edge. Implement this algorithm in Matab, by assuming that at each time one edge is picked with uniform probability $p_i = \frac{1}{8}$ and the previous updates are applied. Plot the error from the desired average of all nodes for different Monte-Carlo realizations of the previous algorithm in log-scale, i.e. $\log(\sum_i |x_i(t) - \frac{1}{6}\sum_i \theta_i|^2)$ as a function of t.

EXERCISE 4 : One - dimensional coverage control problem

The goal of this exercise is to deal with the 1-dimensional coverage control problem. Consider a segment $\mathcal{L} = [0, L]$, $L > 0$, and n robots. In this case a partition is given by n segments $[\ell_i, r_i]$, $i = 1, \dots, n$, (ℓ_i and r_i denote, respectively, the left and the right extremes of the segment assigned to robot i) such that

- $\ell_1 = 0$ and $r_n = L$; and
- $r_i = \ell_{i+1}$ for $i = 1, \dots, n-1$.

Assume the density function ϕ to be homogeneous, namely, $\phi(q) = 1/L$ for all $q \in \mathcal{L}$. The student is required to address the following questions.

- Given the n positions of the robot $P = (p_1, \dots, p_n)$, compute the function $H_{V(P)}$ and verify the property in Equation (2) of the *CoverageControl* notes;
- Could you characterize the set of local minima of $H_{V(P)}$?
- Is it possible to compute the Hessian of $H_{V(P)}$?
- Design and simulate a synchronous discrete-time coverage control algorithm;
- Design and simulate a symmetric gossip discrete-time coverage control algorithm (assume that, at each iteration, one pair of neighboring robots is randomly selected);
- Try to rewrite the above algorithms as consensus algorithms (*Hint*: perform a suitable change of variables such as $L_i = r_i - \ell_i$,...)